APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2020 (August)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: August 19, 2020

Time: 9:00 AM – 1:00 PM

Detailed Instructions for Zoom Proctored Qual: CAM Area Exam

1. This exam is conducted **via Zoom on August 19, 2020, from 9:00 am to 1:00 pm EDT**.
2. The entire Zoom meeting and chat messages are being recorded.
3. This is a closed book, closed note exam.
4. Hand calculators (or other computing devices) may not be used during the exam.
5. Zoom: You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
6. Audio should be muted, and video must be kept on during the exam.
7. Your computer webcam must fully show your face; your smartphone camera should show your hands and workspace, with the pages of paper being used for the exam.
8. At the very beginning of the exam, during set up, you will be asked to do a brief “environment scan”, showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
9. You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the “environment scan” on Zoom.
10. You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication with the proctor(s) via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
11. After you finish the exam, scan all of your pages of work, into a single pdf, and email as an attachment to `xiaolin.li@stonybrook.edu` no later than 5 minutes after completion of the exam (i.e., **by 1:05pm, EDT, on Wed, Aug 19**).
12. For each question you answer, start at the top of a clean page of paper and label the problem clearly. If the problem has an associated figure (on the exam page), you can write directly on that figure and include it in the scanned solutions you send by email.
13. Scan and submit your answers in a single pdf file. Make sure you scan carefully, so that images are clear, not blurred or truncated.
14. No students are allowed to leave the Zoom meeting until the exam is over.
15. If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 1:00pm, EDT.
16. After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
17. If your answers are not returned by 1:05 pm, EDT, the exam will not be graded, and a score of zero will be given.
18. If you have a question during the exam, then send a Zoom chat message to the host.

A1. Solve Laplace equation in the domain of annulus

$$\Delta u = 0, \quad \Omega = \{(r,\theta) \in R^2 :\ 1 < r < 2,\ \ 0 < \theta < 2\pi\}$$

with the boundary conditions for $u(r,\theta)$

$$u(1,\theta) = 1,\ \ u(2,\theta) = \theta(2\pi - \theta).$$

The Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

A2. The normalized conservation law for traffic flow is given as

$$u_t + f(u)_x = 0,$$

where $u > 0$ is the traffic density, and $f(u) = u(1-u)$ is the traffic flux.

(a). Find the solution at $t = 1$ with initial condition

$$u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$$

(b). Find the solution at $t = 4$ with initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$$

(c). Find the Riemann solution with the initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}.$$

(d). Find the particular Riemann solution at $x/t = 0$.

A3. Find the number of zeros of $z^4 + 3z^2 + z + 1$ in the unit disc.

A4. Let $n \geq 2$ be an integer. Evaluate the improper integral

$$\int_0^\infty \frac{1}{1+x^n} dx.$$

A5. Suppose that $f(z)$ is analytic on $G = \{0 < |z| < 1\}$ (punctured disc) and $|f(z)| < \log(1/|z|)$ on G. Prove that $f(z)$ is identically zero.

B6. If you are given function values f_0, f_1 and the first order derivatives f'_0, f'_1 at $x_0 = 0, x_1 = h$.

(a). Find the polynomial interpolation $p(x)$ in this interval using all the conditions. Estimate the interpolation error.

(b). Integrate the polynomial in the interval to get the numerical approximation for $\int_0^h f(x)dx$. What is the order of truncation error for the integration?

(c). Find the numerical integral of $\int_a^b f(x)dx$ by dividing the interval $[a, b]$ into n segments and apply (b) to each segment. What is the order of global truncation error?

B7. Write out the fourth order Taylor expansion method and the classical fourth order Runge-Kutta method to solve the following initial value problem

$$y' = y^2, \quad y(x_0) = y_0.$$

Compare the efficiency of the two methods for each step.

B8.

(a) Derive the Dufort-Frankel scheme for the one-dimensional diffusion equation $v_t = \nu v_{xx}$ by starting with the leapfrog scheme for this PDE and replacing the term u_k^n with the time-averaging $(u_k^{n+1} + u_k^{n-1})/2$.

(b) Using the discrete Fourier transform, perform stability analysis of the Dufort-Frankel scheme.

B9. Consider the following initial-value problem for the diffusion equation in two-dimensional space

$$\begin{aligned} v_t &= \nu(v_{xx} + v_{yy}), \quad (x, y) \in (-\infty, \infty) \times (-\infty, \infty),\, t > 0 \\ v(x, y, 0) &= f(x, y). \end{aligned}$$

and a locally one-dimensional scheme

$$\begin{aligned} (1 - r_x\delta_x^2)u_{jk}^{n+1/2} &= u_{jk}^n, \\ (1 - r_y\delta_y^2)u_{jk}^{n+1} &= u_{jk}^{n+1/2}, \end{aligned}$$

where $r_x = \nu\Delta t/\Delta x^2$, $\delta_x^2 u_{jk} = u_{j-1,k} - 2u_{jk} + u_{j+1,k}$, and $r_y = \nu\Delta t/\Delta y^2$, $\delta_y^2 u_{jk} = u_{j,k-1} - 2u_{jk} + u_{j,k+1}$.

(a) Discuss the consistency of this scheme (you may explore a relation of this scheme to the BTCS scheme).

(b) Perform analysis of stability.

B10. Consider a system of linear hyperbolic equations

$$U_t + AU_x + BU_y = 0.$$

(a) Show that the simple splitting

$$e^{-tB\partial_y} e^{-tA\partial_x},$$

where $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y \equiv \frac{\partial}{\partial y}$, is 1st order accurate in t.

(b) Prove that the Strang splitting

$$e^{-\frac{t}{2}A\partial_x} e^{-tB\partial_y} e^{-\frac{1}{2}A\partial_x}$$

is second order accurate in t.